Thermal Network Analysis with TN Solver
Steady Conduction - The Composite Wall Problem

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Outline

Heat Transfer Analysis using Thermal Networks

▶ Heat Transfer Math Model
  ▶ Steady, plane wall conduction with convection
  ▶ Control volumes and the integral form

▶ Introduction to TNSolver

▶ Composite Wall Problem
Heat Transfer
Math Model
Conduction, Convection and Radiation

Help! The barn is on fire.

Figure borrowed from [LL12].
Heat Transfer in Industry
Math Model

Automotive

Electronics Packaging

Aircraft

Aerospace
Heat Transfer in Industry
Math Model

HVAC

Energy Production

Food Production

Naval
Heat Transfer Analysis
Math Model

Answering design questions about thermal energy and temperature

- Hand calculation - back-of-the-envelope
  - On the order of 1-10 equations
- Spreadsheet style
  - Interactive Heat Transfer (IHT 4.0), see p. ix in [BLID11]
  - LibreOffice Calc, Microsoft Excel, MathCAD
- Thermal network or lumped parameter approach
  - On the order of 10-1,000 equations
- Continuum approach - solid model/mesh generation
  - On the order of 1,000-1,000,000 equations
  - Finite Volume Method (FVM)
  - Finite Element Method (FEM)

See Section 1.5, page 38, in [BLID11]
Commercial Thermal Network Solvers
Math Model

- C&R Technologies
  - SINDA/FLUINT, Thermal Desktop, RadCAD
- MSC Software
  - Sinda, SindaRad, Patran
- ESATAN-TMS
  - Thermal, Radiative, CADbench
The Control Volume Concept

Math Model

\[ \sum \text{Energy In} - \sum \text{Energy Out} = \text{Energy Stored, Generated and/or Consumed} \]

Heat (transfer) is thermal energy transfer due to a temperature difference
The steady conduction equation, in Cartesian tensor integral form, is:

$$\int_A q_i n_i \, dA = \int_V \dot{q} \, dV$$

where $\dot{q}$ is a volumetric source and Fourier’s Law of Heat Conduction provides a constitutive model for the heat flux as a function of temperature gradient:

$$q_i = -k \frac{\partial T}{\partial x_i}$$

where $k$ is the isotropic thermal conductivity.
Convection heat transfer from the surface of the control volume is modeled by:

\[ \int_{\Gamma_c} \mathbf{q}_i n_i \, dA = \int_{\Gamma_c} h(T_s - T_c) \, dA, \quad \text{where} \quad \begin{cases} T_s > T_c, & \text{cooling} \\ T_s < T_c, & \text{heating} \end{cases} \]

The convection coefficient, \( h(x_i, t, T_s, T_c) \), is usually a function of position, time, surface temperature, \( T_s \), free stream or bulk temperature, \( T_c \), and other parameters. The value of the coefficient is often evaluated using a correlation.
Introducing TNSolver

TNSolver User Guide

- Thermal Network Solver - TNSolver
- MATLAB/Octave program
  - GNU Octave is an open source clone of MATLAB
- Thermal model is described in a text input file
  - Do not use a word processor, use a text editor, such as:
    - Cross-platform: vim/gvim, emacs, Bluefish, among many others
    - Windows: notepad, Notepad++
    - MacOS: TextEdit, Smultron
    - Linux: see cross-platform options
- Simulation results are both returned from the function and written to text output files for post-processing
Thermal Network Terminology

TNSolver User Guide

- Time dependency
  - Steady state or transient
  - Initial condition is required for transient

- Geometry
  - Control Volume - volume, $V = \int_V dV$
    - Node: ●, $T_{node} = \int_V T(x) dV$
  - Control Volume Surface - area, $A = \int_A dA$
    - Surface Node: ○, $T_{surface \ node} = \int_A T(x) dA$

- Material properties

- Conductors
  - Conduction
  - Convection
  - Radiation

- Boundary conditions
  - Boundary node: ▲

- Sources/sinks
! Simple Wall Model

Begin Solution Parameters
  type = steady
End Solution Parameters

Begin Conductors
  wall conduction in out 2.3 1.2 1.0 ! k L A
  fluid convection out Tinf 2.3 1.0 ! h A
End Conductors

Begin Boundary Conditions
  fixed_T 21.0 in ! Inner wall T
  fixed_T 5.0 Tinf ! Fluid T
End Boundary Conditions

! begins a comment (MATLAB uses %)
Begin Solution Parameters

    title = A thermal network model
    type  = steady ! <steady|transient>
    units = SI    ! <SI|US>

End Solution Parameters
Conduction: Cartesian (The Plane Wall)

The rate of heat transfer, $Q_{ij}$, due to conduction, between the two temperatures $T_i$ and $T_j$, separated by a distance $L$ and area $A$, is:

$$Q_{ij} = \frac{kA}{L} (T_i - T_j)$$

The heat flux, $q_{ij}$, is:

$$q_{ij} = \frac{Q_{ij}}{A} = \frac{k}{L} (T_i - T_j)$$

Begin Conductors

! label type node i node j parameters
  name conduction label label x.x x.x x.x ! k L A

End Conductors
The rate of heat transfer due to convection is:

\[ Q_{ij} = hA(T_i - T_j) \]

```
Begin Conductors

! label type node i node j parameters
  name convection label label x.x x.x ! h A

End Conductors
```
The node temperature, \( T_b \), is specified:

```
Begin Boundary Conditions

! type parameter(s) node(s)
  fixed_T T_b label

End Boundary Conditions
```
Description of the Composite Wall Problem

Consider a composite wall:

See Figure 3.3, on page 117, in [BLID11].
Model Parameters
Composite Wall Model

The inner wall temperature $T_{in} = 1$

The outer wall temperature $T_{out} = 0$

<table>
<thead>
<tr>
<th>Region</th>
<th>Conductivity, $k$</th>
<th>Length, $L$</th>
<th>Area, $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>F</td>
<td>2.0</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>G</td>
<td>$0.001 \leq k_G \leq 2.0$</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>H</td>
<td>3.0</td>
<td>1.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>
There are four control volumes:
Compare with Figure 3.3 (a), on page 117, in [BLID11].
TNSolver Input File for $k_G = 2.0$

Composite Wall Model 1

! Composite wall model: Approach 1 series-parallel

Begin Solution Parameters

type = steady

End Solution Parameters

Begin Conductors

! label type node 1 node 2 parameters
100 conduction Tin 1 1.0 1.0 2.0 ! k_E L_E A_E
101 conduction 1 2 2.0 2.0 1.0 ! k_F L_F A_F
102 conduction 1 2 2.0 2.0 1.0 ! k_G L_G A_G
103 conduction 2 Tout 3.0 1.0 2.0 ! k_H L_H A_H

End Conductors

Begin Boundary Conditions

! type parameter(s) node(s)
fixed_T 1.0 Tin ! inner wall temperature
fixed_T 0.0 Tout ! outer wall temperature

End Boundary Conditions
TNSolver Output for \( k_G = 2.0 \)

Composite Wall Model 1

### Nodes

<table>
<thead>
<tr>
<th>Label</th>
<th>Material</th>
<th>Volume (m^3)</th>
<th>Temperature (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tin</td>
<td>N/A</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>N/A</td>
<td>0</td>
<td>0.571429</td>
</tr>
<tr>
<td>2</td>
<td>N/A</td>
<td>0</td>
<td>0.142857</td>
</tr>
<tr>
<td>Tout</td>
<td>N/A</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Conductors

<table>
<thead>
<tr>
<th>Label</th>
<th>Type</th>
<th>Node i</th>
<th>Node j</th>
<th>( Q_{ij} ) (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>conduction</td>
<td>Tin</td>
<td>1</td>
<td>0.857143</td>
</tr>
<tr>
<td>101</td>
<td>conduction</td>
<td>1</td>
<td>2</td>
<td>0.428571</td>
</tr>
<tr>
<td>102</td>
<td>conduction</td>
<td>1</td>
<td>2</td>
<td>0.428571</td>
</tr>
<tr>
<td>103</td>
<td>conduction</td>
<td>2</td>
<td>Tout</td>
<td>0.857143</td>
</tr>
</tbody>
</table>
Verification of the Results
Composite Wall Model 1

Analytical solution is used to compare to TNSolver results

Using Equation (3.19), page 116, in [BLID11]:

\[
R_{tot} = \frac{1}{UA} = \frac{L_E}{k_E A_E} + \left[ \frac{k_F A_F}{L_F} + \frac{k_G A_G}{L_G} \right]^{-1} + \frac{L_H}{k_H A_H}
\]

\[
R_{tot} = \frac{1}{UA} = \frac{1}{(1)(2)} + \left[ \frac{(2)(1)}{2} + \frac{(2)(1)}{2} \right]^{-1} + \frac{1}{(3)(2)} = \frac{7}{6} = 1.1667
\]

\[
Q = UA \Delta T = \frac{(T_{in} - T_{out})}{R_{tot}} = \frac{(1.0 - 0.0)}{1.1667} = 0.857
\]
Second Approach
Composite Wall Model 2

There are six control volumes:

- Tin
  - 100
  - Surface node

- Tout
  - 2
  - 101

- Tout
  - 3
  - 102

- Tout
  - 4
  - 103
  - 104

- Tout
  - 5
  - 105
Network Diagram
Composite Wall Model 2

Compare with Figure 3.3 (b), on page 117, in [BLID11].
TNSolver Input File for $k_G = 2.0$

Composite Wall Model 2

! Composite wall model: Approach 2 - parallel conductors

Begin Solution Parameters

  type = steady

End Solution Parameters

Begin Conductors

  ! label type node 1 node 2 parameters
  100 conduction Tin 1 1.0, 1.0, 1.0 ! k_E L_E A_E
  101 conduction 1 2 2.0, 2.0, 1.0 ! k_F L_F A_F
  102 conduction 2 Tout 3.0, 1.0, 1.0 ! k_H L_H A_H
  103 conduction Tin 3 1.0, 1.0, 1.0 ! k_E L_E A_E
  104 conduction 3 4 2.0, 2.0, 1.0 ! k_G L_G A_G
  105 conduction 4 Tin 3.0, 1.0, 1.0 ! k_H L_H A_H

End Conductors

Begin Boundary Conditions

  ! type parameter(s) node(s)
  fixed_T 1.0 Tin ! inner wall temperature
  fixed_T 0.0 Tout ! outer wall temperature

End Boundary Conditions
There are six control volumes:

1. Tin
2. 100
3. 101
4. 106

2. Tin
3. 102
4. 103
5. 107
6. 106

3. Tin
4. 110
5. 111
6. 112
7. 113
8. 114
9. 115
10. 108

○ surface node    ● volume node
TNSolver Input File for $k_G = 2.0$

Composite Wall Model 3

! Composite wall model: Approach 3

Begin Solution Parameters

type = steady

End Solution Parameters

Begin Conductors

! label type node 1 node 2 parameters
100 conduction Tin 1 1.0, 0.5, 1.0 ! k_E L_E A_E/2
101 conduction 1 2 1.0, 0.5, 1.0 ! k_E L_E A_E/2
102 conduction 2 3 2.0, 1.0, 1.0 ! k_F L_F A_F
103 conduction 3 4 2.0, 1.0, 1.0 ! k_F L_F A_F
104 conduction 4 5 3.0, 0.5, 1.0 ! k_H L_H A_H/2
105 conduction 5 Tout 3.0, 0.5, 1.0 ! k_H L_H A_H/2
106 conduction 1 7 1.0, 1.0, 1.0 ! k_E
107 conduction 3 6 2.0, 0.5, 2.0 ! k_F
108 conduction 5 11 3.0, 1.0, 1.0 ! k_H
109 conduction 6 9 2.0, 0.5, 2.0 ! k_G
110 conduction Tin 7 1.0, 0.5, 1.0 ! k_E L_E A_E/2
111 conduction 7 8 1.0, 0.5, 1.0 ! k_E L_E A_E/2
112 conduction 8 9 2.0, 1.0, 1.0 ! k_G L_G A_G
113 conduction 9 10 2.0, 1.0, 1.0 ! k_G L_G A_G
114 conduction 10 11 3.0, 0.5, 1.0 ! k_H L_H A_H/2
115 conduction 11 Tout 3.0, 0.5, 1.0 ! k_H L_H A_H/2

End Conductors

Begin Boundary Conditions

! type parameter(s) node(s)
fixed_T 1.0 Tin ! inner wall temperature
fixed_T 0.0 Tout ! outer wall temperature

End Boundary Conditions
Total Heat Transfer over the Range of $k_G$

Composite Wall Model Summary
Conclusion

- Heat Transfer Analysis in Industry
- Thermal Network Analysis Method
  - Open source TNSolver for Octave/MATLAB
  - Steady, Cartesian conduction and convection conductors
- The Composite Wall Problem
  - Three control volume approaches
  - Overall total heat flow, \( Q \), for \( 0.001 \leq k_G \leq 2.0 \)

Questions?
Obtaining GNU Octave

GNUn Octave

- GNU Octave
  - http://www.gnu.org/software/octave/
- Octave Wiki
  - http://wiki.octave.org
- Octave-Forge Packages (similar to MATLAB Toolbox packages)
- For Windows installation I would suggest the MinGW installation.
  - If you already have Cygwin installed, then install that version.
## SI Units

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Fundamental</th>
<th>Derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>$m$</td>
<td>$M$</td>
<td>$kg$</td>
</tr>
<tr>
<td>Length</td>
<td>$x, y, z$</td>
<td>$L$</td>
<td>$m$</td>
</tr>
<tr>
<td>Area</td>
<td>$A$</td>
<td>$L^2$</td>
<td>$m^2$</td>
</tr>
<tr>
<td>Volume</td>
<td>$V$</td>
<td>$L^3$</td>
<td>$m^3$</td>
</tr>
<tr>
<td>Time</td>
<td>$t$</td>
<td>$t$</td>
<td>$s$</td>
</tr>
<tr>
<td>Force</td>
<td>$F$</td>
<td>$M \cdot L \cdot t^2$</td>
<td>$kg \cdot m \cdot s^{-2}$</td>
</tr>
<tr>
<td>Energy</td>
<td>$E$</td>
<td>$M \cdot L^2 \cdot t^2$</td>
<td>$kg \cdot m^2 \cdot s^{-2}$</td>
</tr>
<tr>
<td>Power</td>
<td>$P$</td>
<td>$M \cdot L^2 \cdot t^3$</td>
<td>$kg \cdot m^2 \cdot s^{-3}$</td>
</tr>
<tr>
<td>Rate of heat transfer</td>
<td>$Q = qA$</td>
<td>$M \cdot L^2 \cdot t^3$</td>
<td>$kg \cdot m^2 \cdot s^{-3}$</td>
</tr>
<tr>
<td>Heat flux</td>
<td>$q$</td>
<td>$M \cdot t^3$</td>
<td>$kg \cdot s^{-3}$</td>
</tr>
<tr>
<td>Heat generation rate per unit volume</td>
<td>$\dot{q}$</td>
<td>$L \cdot t^3$</td>
<td>$m \cdot s^{-3}$</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T$</td>
<td>$T$</td>
<td>$K$</td>
</tr>
<tr>
<td>Pressure</td>
<td>$P$</td>
<td>$M \cdot L \cdot t^2$</td>
<td>$kg \cdot m \cdot s^{-2}$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$u, v, w$</td>
<td>$L \cdot t^{-1}$</td>
<td>$m \cdot s^{-1}$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$M \cdot t^{-3}$</td>
<td>$kg \cdot m^{-3}$</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$k$</td>
<td>$M \cdot L \cdot t^{-3} \cdot T$</td>
<td>$kg \cdot m \cdot s^{-2} \cdot K$</td>
</tr>
<tr>
<td>Specific heat</td>
<td>$c$</td>
<td>$L^2 \cdot t^{-2}$</td>
<td>$m^2 \cdot s^{-2} \cdot K$</td>
</tr>
<tr>
<td>Dynamic (absolute) viscosity</td>
<td>$\mu$</td>
<td>$M \cdot L \cdot t^{-1}$</td>
<td>$kg \cdot m \cdot s^{-1}$</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>$\alpha$</td>
<td>$M \cdot L \cdot t^{-1}$</td>
<td>$kg \cdot m \cdot s^{-1}$</td>
</tr>
<tr>
<td>Kinematic Viscosity</td>
<td>$\nu$</td>
<td>$M \cdot L \cdot t^{-1}$</td>
<td>$kg \cdot m \cdot s^{-1}$</td>
</tr>
<tr>
<td>Convective heat transfer coefficient</td>
<td>$h$</td>
<td>$M \cdot L \cdot t^{-3} \cdot T$</td>
<td>$kg \cdot s^{-3} \cdot K$</td>
</tr>
</tbody>
</table>
Cartesian Tensor Notation (Einstein Convention)

Cartesian tensor notation is a compact method for writing equations. A few simple rules can be used to expand an equation into its full form based on the subscript indices. The range of the indices are based on the spatial dimension of the problem. If an index is repeated within a term of the equation, then a summation over the index is implied.

Two-dimensions:

\[ q_i n_i = q_1 n_1 + q_2 n_2 = q_x n_x + q_y n_y \]

Three-dimensions:

\[ q_i n_i = q_1 n_1 + q_2 n_2 + q_3 n_3 = q_x n_x + q_y n_y + q_z n_z \]
References I

*Introduction to Heat Transfer.*  

*A Heat Transfer Textbook.*  
Available at: http://ahtt.mit.edu.